Modeling and numerical analysis of coupled circuit and electromagnetic simulation

Minisymposia: Integrated Circuit/EM Simulation and Design Technologies for Radio Systems II

Sascha Baumanns   Michael Matthes

Mathematical Institute
University of Cologne

The 16-th European Conference on Mathematics for Industry
July 26-30, 2010 Wuppertal, Germany

Outline

1. DAE formulation
2. Circuit-Device Model
3. Structural Properties
4. Outlook

DAE formulation

Regarding

$$A(y,t)\frac{\mathrm{d}}{\mathrm{d}t}y + b(y,t) = 0,$$

where $A(y,t)$ is singular for all $(y,t) \in \mathcal{D} \times \mathcal{I}$

DAE formulation

Regarding

$$A(y,t)\frac{\mathrm{d}}{\mathrm{d}t}d\,(y,t) + b\,(y,t) = 0,$$

where $A(y,t)$ is singular for all $(y,t) \in \mathcal{D} \times \mathcal{I}$

DAE formulation

Regarding

$$A(y,t)\frac{\mathrm{d}}{\mathrm{d}t}d\,(y,t) + b\,(y,t) = 0,$$

where $A(y,t)$ is singular for all $(y,t) \in \mathcal{D} \times \mathcal{I}$ and A, d, b: $\mathcal{D} \times \mathcal{I} \to \mathbb{R}^n$ are continuous with continuous partial derivatives $A_y\,(\cdot)$, $d_y\,(\cdot)$, $b_y\,(\cdot)$.

DAE formulation

Regarding

$$A(y,t)\frac{\mathrm{d}}{\mathrm{d}t}d\left(y,t\right)+b\left(y,t\right)=0,$$

where $A(y,t)$ is singular for all $(y,t)\in\mathcal{D}\times\mathcal{I}$ and A, d, b: $\mathcal{D}\times\mathcal{I}\to\mathbb{R}^n$ are continuous with continuous partial derivatives $A_y\left(\cdot\right)$, $d_y\left(\cdot\right)$, $b_y\left(\cdot\right)$.

We speak of a properly stated leading term if

$$\ker A\left(\cdot\right)\oplus\operatorname{im}d_y\left(\cdot\right)=\mathbb{R}^n$$

and there is a projector $R\left(t\right)\in\mathcal{C}^1\left(\mathcal{I},\mathbb{R}^n\right)$ with

$$\ker A\left(\cdot\right)=\ker R\left(t\right)\;\text{ and }\;\operatorname{im}d_y\left(\cdot\right)=\operatorname{im}R\left(t\right).$$

Properly stated leading term

- Introduced in [März 2002]

Properly stated leading term

- Introduced in [März 2002]
- DAEs are mostly not formulated in that form

Properly stated leading term

- Introduced in [März 2002]
- DAEs are mostly not formulated in that form
- reformulation not always possible

Properly stated leading term

- Introduced in [März 2002]
- DAEs are mostly not formulated in that form
- reformulation not always possible
- leading term $d(\cdot)$ figures out which derivatives are involved

Properly stated leading term

- Introduced in [März 2002]
- DAEs are mostly not formulated in that form
- reformulation not always possible
- leading term $d(\cdot)$ figures out which derivatives are involved
- for a class of DAEs the BDF and RK methods are stability preserving [Higueras, März, Tischendorf 2003].

ODE vs. DAE

- implicit ordinary differential equations

ODE vs. DAE

- implicit ordinary differential equations
- some solution components are determined by constraints

ODE vs. DAE

- implicit ordinary differential equations
- some solution components are determined by constraints
- restriction of initial values

ODE vs. DAE

- implicit ordinary differential equations
- some solution components are determined by constraints
- restriction of initial values
- not only integration problems but also differentiation problems

ODE vs. DAE

- implicit ordinary differential equations
- some solution components are determined by constraints
- restriction of initial values
- not only integration problems but also differentiation problems
- parts of the DAE must be sufficiently differentiable

Index

- DAE is characterized by its index

Index

- DAE is characterized by its index
- measure of the equation's sensitivity

Index

- DAE is characterized by its index
- measure of the equation's sensitivity
- there are different index concepts

Index

- DAE is characterized by its index
- measure of the equation's sensitivity
- there are different index concepts
- in the linear case different index concepts let to the same index

Index

- DAE is characterized by its index
- measure of the equation's sensitivity
- there are different index concepts
- in the linear case different index concepts let to the same index
- using tractability index

Index

- DAE is characterized by its index
- measure of the equation's sensitivity
- there are different index concepts
- in the linear case different index concepts let to the same index
- using tractability index
- require low smoothness and is algorithmic

Modified Nodal Analysis

$$A_C \frac{\mathrm{d}}{\mathrm{d}t} q(A_C^\top e, t) + A_R g(A_R^\top e, t) + A_L j_L + A_V j_V + A_I i_s(t) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(j_L, t) - A_L^\top e = 0,$$
$$A_V^\top e - v_s(t) = 0,$$

where

- $v_s(t)$, $i_s(t)$, $q(\cdot)$, $g(\cdot)$, $\phi(\cdot)$ given functions,

Modified Nodal Analysis

$$
\begin{aligned}
A_C \frac{\mathrm{d}}{\mathrm{d}t} q(A_C^\top e, t) + A_R g(A_R^\top e, t) + A_L j_L + A_V j_V + A_I i_s(t) &= 0,\\
\frac{\mathrm{d}}{\mathrm{d}t}\phi(j_L, t) - A_L^\top e &= 0,\\
A_V^\top e - v_s(t) &= 0,
\end{aligned}
$$

where

- $v_s(t)$, $i_s(t)$, $q(\cdot)$, $g(\cdot)$, $\phi(\cdot)$ given functions,
- incidence matrices $[A_V\ A_I\ A_C\ A_R\ A_L]$.

Maxwell's equation

Law's

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial}{\partial t} B$$

$$\nabla \times H = J_c + \frac{\partial}{\partial t} D$$

Maxwell's equation

Law's

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial}{\partial t} B$$

$$\nabla \times H = J_c + \frac{\partial}{\partial t} D$$

Continuity equation

$$\nabla \cdot J_c + \nabla \cdot \frac{\partial}{\partial t} D = 0$$

Maxwell's equation

Law's

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial}{\partial t} B$$

$$\nabla \times H = J_c + \frac{\partial}{\partial t} D$$

Constitutive relations

$$D = \varepsilon E$$

$$H = \nu B$$

$$J_c = \sigma E$$

Continuity equation

$$\nabla \cdot J_c + \nabla \cdot \frac{\partial}{\partial t} D = 0$$

Maxwell's equation

Law's

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial}{\partial t} B$$

$$\nabla \times H = J_c + \frac{\partial}{\partial t} D$$

Constitutive relations

$$D = \varepsilon E$$

$$H = \nu B$$

$$J_c = \sigma E$$

Boundary conditions

electric: "current gate"

magnetic: "current wall"

Continuity equation

$$\nabla \cdot J_c + \nabla \cdot \frac{\partial}{\partial t} D = 0$$

Potentials

Scalar and vector potential

$$E = -\nabla V - \frac{\partial}{\partial t} A$$

$$B = \nabla \times A$$

Potentials

Scalar and vector potential

$$E = -\nabla V - \frac{\partial}{\partial t} A$$

$$B = \nabla \times A$$

Gauge

$$\nabla \cdot A = 0$$

Potentials

Scalar and vector potential

$$E = -\nabla V - \frac{\partial}{\partial t} A$$

$$B = \nabla \times A$$

Gauge

$$\nabla \cdot A = 0$$

Pseudo-canonical momentum

$$\Pi = \frac{\partial}{\partial t} A$$

Potentials

Scalar and vector potential

$$E = -\nabla V - \frac{\partial}{\partial t} A$$

$$B = \nabla \times A$$

Gauge

$$\nabla \cdot A = 0$$

Pseudo-canonical momentum

$$\Pi = \frac{\partial}{\partial t} A$$

Potential formulation

$$\frac{\partial}{\partial t} \left(\nabla \cdot \varepsilon \nabla V + \nabla \cdot \sigma A + \nabla \cdot \varepsilon \Pi \right) + \nabla \cdot \sigma \nabla V = 0$$

$$\frac{\partial}{\partial t} \left(\varepsilon \nabla V + \sigma A + \varepsilon \Pi \right) + \sigma \nabla V + \nabla \times \nu \nabla \times A - \nu \nabla \nabla \cdot A = 0$$

$$\frac{\partial}{\partial t} A - \Pi = 0$$

FIT space discretization

$$\frac{\partial}{\partial t}\left(\nabla \cdot \varepsilon \nabla V + \nabla \cdot \sigma A + \nabla \cdot \varepsilon \Pi\right) + \nabla \cdot \sigma \nabla V = 0$$

$$\frac{\partial}{\partial t}\left(\varepsilon \nabla V + \sigma A + \varepsilon \Pi\right) + \sigma \nabla V + \nabla \times \nu \nabla \times A - \nu \nabla \nabla \cdot A = 0$$

$$\frac{\partial}{\partial t} A - \Pi = 0$$

FIT space discretization

$$\frac{\partial}{\partial t}\left(\nabla \cdot \varepsilon \nabla V + \nabla \cdot \sigma A + \nabla \cdot \varepsilon \Pi\right) + \nabla \cdot \sigma \nabla V = 0$$

$$\frac{\partial}{\partial t}\left(\varepsilon \nabla V + \sigma A + \varepsilon \Pi\right) + \sigma \nabla V + \nabla \times \nu \nabla \times A - \nu \nabla \nabla \cdot A = 0$$

$$\frac{\partial}{\partial t} A - \Pi = 0$$

- consistent formulation for the discrete Maxwell's equations

FIT space discretization

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\nabla \cdot \varepsilon \nabla V + \nabla \cdot \sigma A + \nabla \cdot \varepsilon \Pi\right) + \nabla \cdot \sigma \nabla V &= 0 \\
\frac{\partial}{\partial t}\left(\varepsilon \nabla V + \sigma A + \varepsilon \Pi\right) + \sigma \nabla V + \nabla \times \nu \nabla \times A - \nu \nabla \nabla \cdot A &= 0 \\
\frac{\partial}{\partial t} A - \Pi &= 0
\end{aligned}
$$

- consistent formulation for the discrete Maxwell's equations
- error only in the constitutive relations

FIT space discretization

$$\frac{\partial}{\partial t}\left(\nabla\cdot\varepsilon\nabla V + \nabla\cdot\sigma A + \nabla\cdot\varepsilon\Pi\right) + \nabla\cdot\sigma\nabla V = 0$$
$$\frac{\partial}{\partial t}\left(\varepsilon\nabla V + \sigma A + \varepsilon\Pi\right) + \sigma\nabla V + \nabla\times\nu\nabla\times A - \nu\nabla\nabla\cdot A = 0$$
$$\frac{\partial}{\partial t}A - \Pi = 0$$

- consistent formulation for the discrete Maxwell's equations
- error only in the constitutive relations
- discr. operators contains only topological information

FIT space discretization

$$\frac{\partial}{\partial t}\left(\nabla \cdot \varepsilon \nabla V + \nabla \cdot \sigma A + \nabla \cdot \varepsilon \Pi\right) + \nabla \cdot \sigma \nabla V = 0$$

$$\frac{\partial}{\partial t}\left(\varepsilon \nabla V + \sigma A + \varepsilon \Pi\right) + \sigma \nabla V + \nabla \times \nu \nabla \times A - \nu \nabla \nabla \cdot A = 0$$

$$\frac{\partial}{\partial t} A - \Pi = 0$$

- consistent formulation for the discrete Maxwell's equations
- error only in the constitutive relations
- discr. operators contains only topological information
- need two grids

FIT space discretization

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{\mathbf{S}}\mathbf{M}_\varepsilon \mathbf{G} v + \widetilde{\mathbf{S}}\mathbf{M}_\sigma a + \widetilde{\mathbf{S}}\mathbf{M}_\varepsilon \pi\right) + \widetilde{\mathbf{S}}\mathbf{M}_\sigma \mathbf{G} v = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{M}_\varepsilon \mathbf{G} v + \mathbf{M}_\sigma a + \mathbf{M}_\varepsilon \pi\right) + \mathbf{M}_\sigma \mathbf{G} v + \widetilde{\mathbf{C}}\mathbf{M}_\nu \mathbf{C} a - \mathbf{M}_\nu \mathbf{G}\mathbf{S} a = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} a - \pi = 0$$

- $V \to v$, $A \to a$, $\Pi \to \pi$, $\varepsilon \to \mathbf{M}_\varepsilon$, $\sigma \to \mathbf{M}_\sigma$ and $\nu \to \mathbf{M}_\nu$

FIT space discretization

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{\mathbf{S}}\mathbf{M}_\varepsilon\mathbf{G}v + \widetilde{\mathbf{S}}\mathbf{M}_\sigma a + \widetilde{\mathbf{S}}\mathbf{M}_\varepsilon\pi\right) + \widetilde{\mathbf{S}}\mathbf{M}_\sigma\mathbf{G}v = 0$$

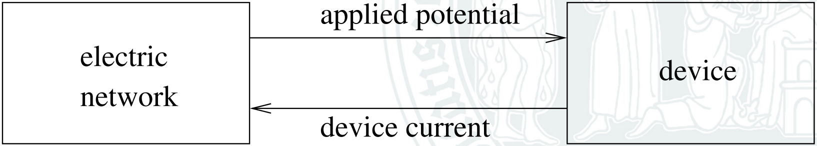
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{M}_\varepsilon\mathbf{G}v + \mathbf{M}_\sigma a + \mathbf{M}_\varepsilon\pi\right) + \mathbf{M}_\sigma\mathbf{G}v + \widetilde{\mathbf{C}}\mathbf{M}_\nu\mathbf{C}a - \mathbf{M}_\nu\mathbf{G}\mathbf{S}a = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}a - \pi = 0$$

- $V \to v$, $A \to a$, $\Pi \to \pi$, $\varepsilon \to \mathbf{M}_\varepsilon$, $\sigma \to \mathbf{M}_\sigma$ and $\nu \to \mathbf{M}_\nu$
- $\nabla \to \mathbf{G}$, $\widetilde{\mathbf{G}}$, $\nabla\times \to \mathbf{C}$, $\widetilde{\mathbf{C}}$ and $\nabla\cdot \to \mathbf{S}$, $\widetilde{\mathbf{S}}$

Coupling Conditions

Coupling via boundary conditions and source term.

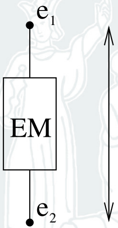


Coupling Conditions

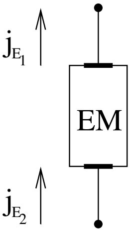
Dirichlet boundary condition for V:

$$V = V\left(x, A_E^\top e\right)$$

where A_E describes the circuit topology and $A_E^\top e$ the potential difference.



Coupling Conditions



Current through the k-th contact:

$$j_{E_k} = \int_{\Gamma_k} J_t \cdot \nu \, \mathrm{d}s$$

where $J_t = r(V, A, \Pi)$ is linear.

Coupled System

$$A_C \frac{\mathrm{d}}{\mathrm{d}t} q(A_C^\top e, t) + A_R g(A_R^\top e, t) + A_L j_L + A_V j_V + A_I i_s(t) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(j_L, t) - A_L^\top e = 0,$$
$$A_V^\top e - v_s(t) = 0$$

where

- $v_s(t)$, $i_s(t)$, $q(\cdot)$, $g(\cdot)$, $\phi(\cdot)$ given functions,
- incidence matrices $[A_V \; A_I \; A_C \; A_R \; A_L \quad]$

Coupled System

$$
\begin{aligned}
A_C \frac{\mathrm{d}}{\mathrm{d}t} q(A_C^\top e, t) + A_R g(A_R^\top e, t) + A_L j_L + A_V j_V + A_I i_s(t) + A_E j_E &= 0, \\
\frac{\mathrm{d}}{\mathrm{d}t}\phi(j_L, t) - A_L^\top e &= 0, \\
A_V^\top e - v_s(t) &= 0, \\
j_E + f(A_E^\top \frac{\mathrm{d}e}{\mathrm{d}t}, \frac{\mathrm{d}v}{\mathrm{d}t}, \frac{\mathrm{d}a}{\mathrm{d}t}, \frac{\mathrm{d}\pi}{\mathrm{d}t}, A_E^\top e, v) &= 0, \\
h(A_E^\top \frac{\mathrm{d}e}{\mathrm{d}t}, \frac{\mathrm{d}v}{\mathrm{d}t}, \frac{\mathrm{d}a}{\mathrm{d}t}, \frac{\mathrm{d}\pi}{\mathrm{d}t}, A_E^\top e, v, a, \pi) &= 0
\end{aligned}
$$

where

- $v_s(t)$, $i_s(t)$, $q(\cdot)$, $g(\cdot)$, $\phi(\cdot)$ given functions,
- incidence matrices $[A_V\ A_I\ A_C\ A_R\ A_L\ A_E]$
- $f(\cdot)$, $h(\cdot)$ linear functions from the space discretization

Coupled System

$$A_C \frac{\mathrm{d}}{\mathrm{d}t} q(A_C^\top e, t) + A_R g(A_R^\top e, t) + A_L j_L + A_V j_V + A_I i_s(t) + A_E j_E = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi(j_L, t) - A_L^\top e = 0,$$

$$A_V^\top e - v_s(t) = 0,$$

$$j_E + f(A_E^\top \frac{\mathrm{d}e}{\mathrm{d}t}, \frac{\mathrm{d}v}{\mathrm{d}t}, \frac{\mathrm{d}a}{\mathrm{d}t}, \frac{\mathrm{d}\pi}{\mathrm{d}t}, A_E^\top e, v) = 0,$$

$$h(A_E^\top \frac{\mathrm{d}e}{\mathrm{d}t}, \frac{\mathrm{d}v}{\mathrm{d}t}, \frac{\mathrm{d}a}{\mathrm{d}t}, \frac{\mathrm{d}\pi}{\mathrm{d}t}, A_E^\top e, v, a, \pi) = 0$$

where

- $v_s(t)$, $i_s(t)$, $q(\cdot)$, $g(\cdot)$, $\phi(\cdot)$ given functions,
- incidence matrices $[A_V \, A_I \, A_C \, A_R \, A_L \, A_E]$
- $f(\cdot)$, $h(\cdot)$ linear functions from the space discretization

Physical reasons and passivity

- A_V and $[A_V \, A_C \, A_R \, A_L]^\top$ have full column rank

Physical reasons and passivity

- A_V and $[A_V\, A_C\, A_R\, A_L]^\top$ have full column rank
- The functions $q\,(u,t)\,, g\,(u,t)$ and $\phi\,(j,t)$ are continuous differentiable with

$$C\,(u,t) := \frac{\partial q\,(u,t)}{\partial u} \qquad G\,(u,t) := \frac{\partial g\,(u,t)}{\partial u}$$

$$L\,(j,t) := \frac{\partial \phi\,(j,t)}{\partial j}$$

positive definite

Index Result

Theorem

The coupled system after space discretization is a DAE of the form

$$A\frac{\mathrm{d}}{\mathrm{d}t}d\,(y,t)+b\,(y,t)=0$$

with properly stated leading term. The DAE has index-2 and is unique solvable.

Starting values

Index-2 components Ty: appears only linearly in the systems

$$A\frac{\mathrm{d}}{\mathrm{d}t}d\left(Uy,t\right)+b\left(Uy,t\right)+BTy=0$$

Starting values

Index-2 components Ty: appears only linearly in the systems

$$A\frac{\mathrm{d}}{\mathrm{d}t}d\left(Uy,t\right)+b\left(Uy,t\right)+BTy=0$$

Starting values:

- Non-consistent (z^0, y^0) fulfilling

$$Az^0+b\left(y^0,t_0\right)=0$$

Starting values

Index-2 components Ty: appears only linearly in the systems

$$A\frac{\mathrm{d}}{\mathrm{d}t}d\left(Uy,t\right)+b\left(Uy,t\right)+BTy=0$$

Starting values:

- Non-consistent (z^0, y^0) fulfilling

$$Az^0+b\left(y^0,t_0\right)=0$$

- one implicit Euler step starting with y^0

Starting values

Index-2 components Ty: appears only linearly in the systems

$$A\frac{\mathrm{d}}{\mathrm{d}t}d\left(Uy,t\right)+b\left(Uy,t\right)+BTy=0$$

Starting values:

- Non-consistent (z^0, y^0) fulfilling

$$Az^0+b\left(y^0,t_0\right)=0$$

- one implicit Euler step starting with y^0
- consistent value y_1 at $t+h$.

Outlook

- consistent value within the first step

Outlook

- consistent value within the first step
- Lorenz instead of Coulomb gauge

Outlook

- consistent value within the first step
- Lorenz instead of Coulomb gauge
- Lorenz vs. Coulomb gauge

Outlook

- consistent value within the first step
- Lorenz instead of Coulomb gauge
- Lorenz vs. Coulomb gauge
- (tractability) index-1 result

Thanks for your attention!