Structural characterization of circuit configurations with undamped oscillations

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ECMI 2010
Wuppertal, July 27, 2010

Outline

1. motivation
2. circuit modeling
3. branch oriented analysis
4. purely imaginary eigenvalues

Motivation

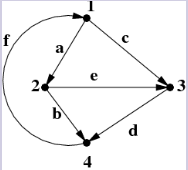
- When do circuits oscillate without damping?
- Which circuit configurations lead to undamped oscillations independently from the parameter values?
- When does a circuit model has purely imaginary eigenvalues?

Circuit Modeling - Incidence Matrix

$A = (a_{ij}) \in \mathbb{R}^{n \times b}$ with

$$a_{ij} := \begin{cases} 1 & \text{if branch } j \text{ leaves node } i \\ -1 & \text{if branch } j \text{ enters node } i \\ 0 & \text{else} \end{cases}$$

Example

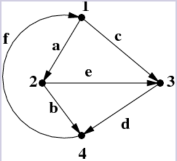


$$A = \begin{array}{ccccccc} \left(\begin{array}{cccccc} 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right) & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \begin{array}{cccccc} a & b & c & d & e & f \end{array} & \end{array}$$

Circuit Modeling - Kirchhoff's laws

- i branch currents (e.g. $i = (i_a, i_b, i_c, i_d, i_e, i_f)$)
- u branch voltages (e.g. $u = (u_a, u_b, u_c, u_d, u_e, u_f)$)
- e nodal potentials (e.g. $e = (e_1, e_2, e_3, e_4)$)

Example



$$A = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{pmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \\ \begin{array}{cccccc} a & b & c & d & e & f \end{array} & \end{array}$$

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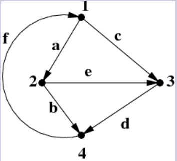
KCL

$$Ai = 0$$

(e.g. $i_a + i_c - i_f = 0$)

Example



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KCL

$$Ai = 0$$

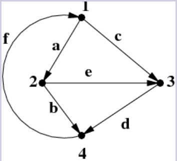
(e.g. $i_a + i_c - i_f = 0$)

KVL

$$A^\top e = u$$

(e.g. $e_1 - e_2 = u_a$)

Example



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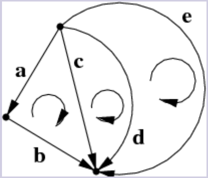
Circuit Modeling - Loop Matrix

$\tilde{B} = (b_{ij}) \in \mathbb{R}^{m \times b}$ with

$$b_{ij} = \begin{cases} 1 & \text{if branch } j \text{ belongs to loop } i \text{ with the same orientation} \\ -1 & \text{if branch } j \text{ belongs to loop } i \text{ with different orientation} \\ 0 & \text{else} \end{cases}$$

Example



$$\tilde{B} \quad = \quad \begin{array}{cc} \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} & \begin{array}{c} abc \\ cd \\ de \\ abd \\ abe \\ ce \end{array} \\ \begin{array}{ccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} \end{array} & \end{array}$$

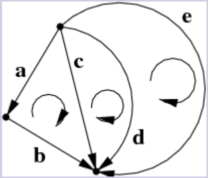
Circuit Modeling - Loop Matrix

KVL

$$\tilde{B}u = 0$$

$$(\text{e.g. } -u_a - u_b + u_c = 0)$$

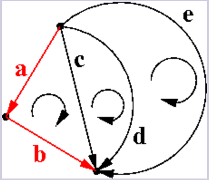
Example



$$\tilde{B} \quad = \quad \begin{array}{cc} \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix} & \begin{array}{c} abc \\ cd \\ de \\ abd \\ abe \\ ce \end{array} \\ \begin{array}{ccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} \end{array} & \end{array}$$

Circuit Modeling - Fundamental Loop Matrix

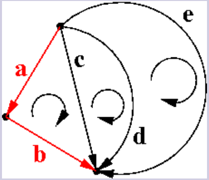
Example



$$\tilde{B} = \begin{array}{cccccc} -1 & -1 & 1 & 0 & 0 & abc \\ 0 & 0 & -1 & 1 & 0 & cd \\ 0 & 0 & 0 & -1 & 1 & de \\ -1 & -1 & 0 & 1 & 0 & abd \\ -1 & -1 & 0 & 0 & 1 & abe \\ 0 & 0 & -1 & 0 & 1 & ce \\ \text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \end{array}$$

Circuit Modeling - Fundamental Loop Matrix

Example



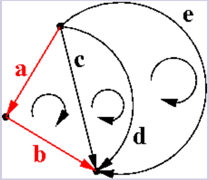
$$B = \begin{array}{cc} \left(\begin{array}{rrrrr} -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{array}\right) & \begin{array}{l} abc \\ abd \\ abe \end{array} \\ \begin{array}{ccccc} a & b & c & d & e \end{array} & \end{array}$$

Circuit Modeling - Fundamental Loop Matrix

KVL

$$Bu = 0$$

Example



$$B = \begin{array}{cc} \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} abc \\ abd \\ abe \end{array} \\ \begin{array}{ccccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} \end{array} & \end{array}$$

Circuit Modeling - Fundamental Loop Matrix

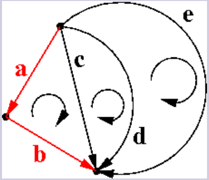
KVL

$$Bu = 0$$

Structure

$$B = (B_\tau \;\; I)$$

Example



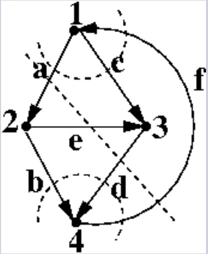
$$B = \begin{array}{cc} \begin{pmatrix} -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} abc \\ abd \\ abe \end{array} \\ \begin{array}{ccccc} a & b & c & d & e \end{array} & \end{array}$$

Circuit Modeling - Cutset Matrix

$\tilde{Q} = (q_{ij}) \in \mathbb{R}^{l \times b}$ with

$$q_{ij} = \begin{cases} 1 & \text{if branch } j \text{ belongs to cutset } i \text{ with the same orientation} \\ -1 & \text{if branch } j \text{ belongs to cutset } i \text{ with different orientation} \\ 0 & \text{else} \end{cases}$$

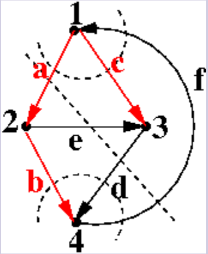
Example



$$\tilde{Q} = \begin{array}{rrrrrrl} 1 & 0 & 0 & 1 & -1 & -1 & aedf \\ 1 & 0 & 1 & 0 & 0 & -1 & acf \\ 0 & 1 & 0 & 1 & 0 & -1 & bdf \\ 0 & 1 & 1 & 0 & 1 & -1 & becf \\ 0 & 0 & 1 & -1 & 1 & 0 & ced \\ 1 & -1 & 0 & 0 & -1 & 0 & aeb \\ 1 & -1 & 1 & -1 & 0 & 0 & abdc \\ \text{a} & \text{b} & \text{c} & \text{d} & \text{e} & \text{f} & \end{array}$$

Circuit Modeling - Fundamental Cutset Matrix

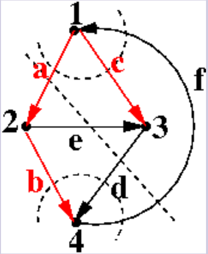
Example



$$\tilde{Q} = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \end{pmatrix} & \begin{array}{c} aedf \\ acf \\ bdf \\ becf \\ ced \\ aeb \\ abdc \end{array} \\ \begin{array}{cccccc} a & b & c & d & e & f \end{array} & \end{array}$$

Circuit Modeling - Fundamental Cutset Matrix

Example



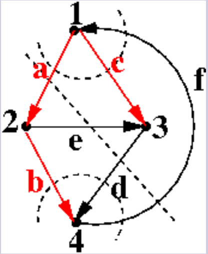
$$Q = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix} & \begin{matrix} aedf \\ bdf \\ ced \end{matrix} \\ \begin{matrix} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} & \mathrm{e} & \mathrm{f} \end{matrix} & \end{array}$$

Circuit Modeling - Fundamental Cutset Matrix

KCL

$$Qi = 0$$

Example



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Circuit Modeling - Fundamental Cutset Matrix

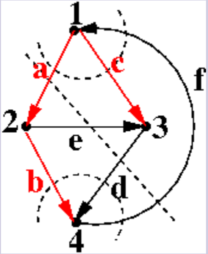
KCL

$$Qi = 0$$

Structure

$$Q = (I \;\; Q_\lambda)$$

Example



$$Q = \begin{array}{ccccccc} \left(\begin{matrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{matrix}\right) & \begin{array}{l} aedf \\ bdf \\ ced \end{array} \\ \begin{matrix} \;a & \;\;b & \;\;c & \;\;d & \;\;e & \;\;f \end{matrix} & \end{array}$$

Circuit Modeling - Network Equations

KCL

$$Ai = 0$$
$$Qi = 0$$

KVL

$$A^\top e = u$$
$$Bu = 0$$

Structural Properties

$$Q = (I \;\; Q_\lambda) \qquad B = (B_\tau \;\; I)$$

$$BQ^\top = 0 \quad \longrightarrow \quad B_\tau = -Q_\lambda^\top$$

Circuit Modeling - Network Equations

KCL

$$Ai = 0$$
$$Qi = 0$$

KVL

$$A^\top e = u$$
$$Bu = 0$$

Basic Circuit Element Equations

- capacitances: $i_C(t) = Cu'_C(t)$
- inductances: $u_L(t) = Li'_L(t)$
- resistances: $i_R(t) = Gu_R(t)$
- current sources: $i_J(t) = j_s(t)$
- voltage sources: $u_V(t) = v_s(t)$

Branch Oriented Analysis

KCL

$$Qi = 0$$

KVL

$$Bu = 0$$

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Branch Oriented Analysis

KCL

$$Q_C i_C + Q_L i_L + Q_R i_R + Q_J i_J + Q_V i_V = 0$$

KVL

$$B_C u_C + B_L u_L + B_R u_R + B_J u_J + B_V u_V = 0$$

Basic Circuit Element Equations

- capacitances: $i_C = Cu'_C$
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- resistances: $i_R = Gu_R$
- current sources: $i_J = j_s(t)$
- voltage sources: $u_V = v_s(t)$

Branch Oriented Analysis

Reduced Branch Oriented System

$$\begin{aligned} Cu_C' &= i_C \\ Li_L' &= u_L \\ 0 &= B_C u_C + B_L u_L + B_R u_R + B_J u_J + B_V v_s(t) \\ 0 &= Q_C i_C + Q_L i_L + Q_R i_R + Q_J j_s(t) + Q_V i_V \\ 0 &= i_R - G u_R \end{aligned}$$

Branch Oriented Analysis

Reduced Branch Oriented System

$$
\begin{aligned}
Cu_C' &= i_C \\
Li_L' &= u_L \\
0 &= B_C u_C + B_L u_L + B_R u_R + B_J u_J + B_V v_s(t) \\
0 &= Q_C i_C + Q_L i_L + Q_R i_R + Q_J j_s(t) + Q_V i_V \\
0 &= i_R - G u_R
\end{aligned}
$$

Matrix Pencil

$$
\begin{pmatrix}
\lambda C & 0 & -I & 0 & 0 & 0 & 0 & 0 \\
0 & -I & 0 & \lambda L & 0 & 0 & 0 & 0 \\
B_C & B_L & 0 & 0 & B_R & 0 & B_J & 0 \\
0 & 0 & Q_C & Q_L & 0 & Q_R & 0 & Q_V \\
0 & 0 & 0 & 0 & G & -I & 0 & 0
\end{pmatrix}
$$

Matrix Pencils

Matrix Pencil (1)

$$\begin{pmatrix} \lambda C & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & \lambda L & 0 & 0 & 0 & 0 \\ B_C & B_L & 0 & 0 & B_R & 0 & B_J & 0 \\ 0 & 0 & Q_C & Q_L & 0 & Q_R & 0 & Q_V \\ 0 & 0 & 0 & 0 & G & -I & 0 & 0 \end{pmatrix}$$

Matrix Pencil (2)

$$\begin{pmatrix} \lambda C & 0 & -I & 0 & 0 & 0 \\ 0 & -I & 0 & \lambda L & 0 & 0 \\ \hat{B}_C & \hat{B}_L & 0 & 0 & \hat{B}_R & 0 \\ 0 & 0 & \hat{Q}_C & \hat{Q}_L & 0 & \hat{Q}_R \\ 0 & 0 & 0 & 0 & G & -I \end{pmatrix}$$

Theorem

Assume that a given circuit has neither V-loops nor I-cutsets. The spectrum of the matrix pencil (1) coincides with that of the pencil (2), provided that the latter is defined by the RLC circuit obtained after open-circuiting current sources and short-circuiting voltage sources.

Purely Imaginary Eigenvalues (PIEs)

Proposition 1

Any eigenvector associated with a PIE verifies $u_R = i_R = 0$.

Proof Idea 1. If $(u, i) = (u_C, u_L, u_R, i_C, i_L, i_R)$ is an eigenvector to a PIE λ then

$$u^* i = (Q^{\mathsf{T}} w)^* i = w^* Q i = 0, \quad \text{since } Bu = 0 \text{ and } \operatorname{Ker} B = \operatorname{Im} Q^{\mathsf{T}}.$$

2. Including the element related equations yields

$$\lambda u_C^* C u_C + \bar{\lambda} i_L^* L i_L + u_R^* G u_R = 0$$

3. Comparing real and imaginary parts concludes the assertion.

Purely Imaginary Eigenvalues (PIEs)

Proposition 2

All eigenvalues of an LC-circuit are PIEs.

Purely Imaginary Eigenvalues (PIEs)

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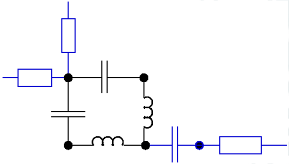
All eigenvalues of an LC-circuit are PIEs.



LC-block

Proposition 3

If the reduced circuit corresponding to pencil (2) has an LC-block then the circuit has a PIE.

The converse of Proposition 3 is not true.

Purely Imaginary Eigenvalues (PIEs)

Theorem

A circuit has a PIE for all positive values of capacitances and inductances if and only if the reduced circuit (after open-circuiting current sources and short-circuiting voltage sources) has an LC-block.

[1] R. Riaza, C. Tischendorf: The hyperbolicity problem in electrical circuit theory. To appear in *Mathematical Methods in The Applied Sciences*, Wiley, 2010.

[2] R. Riaza, C. Tischendorf: Structural characterization of circuit configurations with purely imaginary eigenvalues. Submitted.

Summary

- circuits with undamped oscillations have purely imaginary eigenvalues (PIEs)
- LC-blocks lead always to PIEs
- circuits with PIEs do not necessarily have LC-blocks
- circuits with PIEs for all positive values of capacitances and inductances have LC-blocks (after open-circuiting current sources and short-circuiting voltage sources)