Quasiperiodic steady-state analysis of electronic circuits by a spline basis

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Abstract Multitone Harmonic Balance (HB) is widely used for the simulation of the quasiperiodic steady-state of RF circuits. HB is based on a Fourier expansion of the waveforms. Unfortunately, trigonometric polynomials often exhibit poor convergence properties when the signals are not quasi-sinusoidal, which leads to a prohibitive run-time even for small circuits. Moreover, the approximation of sharp transients leads to the well-known Gibbs phenomenon, which cannot be removed by an increase of the number of Fourier coefficients, because convergence is only guaranteed in the L_2 norm. In this paper we present alternative approaches based on cubic or exponential splines for a periodic or quasiperiodic steady state analysis. Furthermore, it is shown below that the amount of coding effort is negligible if an implementation of HB exists.

1 Introduction: system equations steady states

Depending on the topology of the circuit and the device constitutive equations the Modified Nodal Analysis (MNA) leads to a system of generally nonlinear differential-algebraic equations (DAEs) of first order of dimension N:

$$f(v(t),t) = i(v(t)) + \frac{\mathrm{d}}{\mathrm{d}t} q(v(t)) + b(t) = 0 \tag{1}$$

wherein $t \in \mathbb{R}$ is time and $0 \in \mathbb{R}^N$ the zero vector. Moreover $v : \mathbb{R} \to \mathbb{R}^N$ is the vector of the unknown node voltages and branch currents. $q : \mathbb{R}^N \to \mathbb{R}^N$ is the vector of charges and magnetic fluxes, $i : \mathbb{R}^N \to \mathbb{R}^N$ is the vector of sums of currents

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entering each node and branch voltages. Furthermore $b(t):\mathbb{R}\rightarrow\mathbb{R}^N$ is the vector of input sources.

Let $P(T)$ be the space of all T-periodic $x\in P(T):=\{x\mid x(t)=x(t+T)\}$ and $QP(T_1,T_2,\ldots,T_d)$ the space of all d-quasiperiodic and continuous functions. The Fourier expansions of periodic waveforms is

$$x(t)\in P(T)\Leftrightarrow x(t)=\sum_{k=-\infty}^{\infty}X(k)\cdot e^{j\,k\,\omega_0\,t} \tag{2}$$

and for quasiperiodic signals

$$x(t)=\sum_{k_1=-\infty}^{\infty}\cdots\sum_{k_d=-\infty}^{\infty}X(k_1,\ldots,k_d)\cdot e^{j\,(k_1\,\omega_1+\cdots+k_d\,\omega_d\,)\,t} \tag{3}$$

If the fundamental frequencies are incommensurable, it is guaranteed that they cannot be integer multiples of a common fundamental frequency. In practical applications, the number of fundamental frequencies is $d=2$ to $d=3$.

The input signals or stimuli $b(t)$ are quasiperiodic with typically two or three fundamental frequencies as well. For simplicity only the case $d=2$ is considered here. The extension to an arbitrary number of fundamentals is straightforward.

In [**?**, **?**] it has been shown that a reformulation of the underlying ordinary DAE (**??**) into an appropriate partial DAE system eases the numerical treatment of the multitone problem. This method has been widely accepted by different research groups [**?**, **?**, **?**, **?**, **?**, **?**, **?**].

In [?] the following theorem was proven:

Theorem:

*Consider the system of ordinary differential-algebraic equations (**??**) with quasiperiodic stimulus*

$$b(t)=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}B(k_1,k_2)\cdot e^{jk_1\omega_1 t}\,e^{jk_2\omega_2 t} \tag{4}$$

and the partial DAE system

$$f(\hat{v}(t_1,t_2);t_1,t_2)=i(\hat{v}(t_1,t_2))+\frac{\partial}{\partial t_1}q(\hat{v}(t_1,t_2))+\frac{\partial}{\partial t_2}q(\hat{v}(t_1,t_2))+b(t_1,t_2)=0 \tag{5}$$

where the quasiperiodic stimulus $b(t_1,t_2)$ is given by

$$b(t_1,t_2)=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}B(k_1,k_2)\cdot e^{jk_1\omega_1 t_1}\,e^{jk_2\omega_2 t_2} \tag{6}$$

with Fourier coefficients $B(k_1,k_2)$. Then

$$v(t) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} V(k_1,k_2) \cdot e^{(jk_1\omega_1 + jk_2\omega_2)t} \tag{7}$$

is a steady-state solution of the ordinary differential-algebraic equation (**??**), if and only if

$$\hat{v}(t_1,t_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \hat{V}(k_1,k_2) \cdot e^{jk_1\omega_1 t_1} e^{jk_2\omega_2 t_2} \tag{8}$$

is a steady-state solution of the partial differential-algebraic equation as well. The two solutions are related by $v(t) = \hat{v}(t,t)$ for all $t \in \mathbb{R}$ and the relation $V(k_1,k_2) = \hat{V}(k_1,k_2)$ holds. ■

The theorem states, that a solution of the underlying ordinary DAE can be obtained along a characteristic of the partial DAE.

2 Short summary of the Harmonic Balance method

2.1 HB for periodic steady states

HB approximates the solution in a subspace, which is given by a finite number of Fourier coefficients

$$\mathscr{S} = \left\{ x \,\middle|\, x(t) = \sum_{k=-K}^{K} X(k) \exp\left(j \frac{2\pi k}{T} t \right) \right\} \tag{9}$$

The device constitutive equations are given in most practical cases solely in the time domain, $i(v(t))$ and $q(v(t))$. HB circumvents the implementation problems in the way that the devices are evaluated on an equidistant grid or mesh at collocation points t_i in the time domain. Employing the Discrete Fourier Transform (DFT) or its fast implementation the FFT transforms evaluated waveforms into the frequency domain.

Let $\mathscr{F}$ be the matrix of the DFT, $P := I_N \otimes \mathscr{F}$ a matrix which transforms all N waveforms into the frequency domain and $P^{-1} = I_N \otimes \mathscr{F}^H$ its inverse ($\otimes$ is the Kronecker- or tensor product). The boundary value problem is discretized at $2K+1$ gridpoints t_i and equidistant grid spacing $\Delta t = \frac{T}{2K+1}$. The transformation to and from the spectrum is given in matrix formulation by $X = Px$, $I = Pi$, $Q = Pq$, $B = Pb$. The time derivatives of a waveform are represented in the frequency domain by

$$\bar{\Omega}(\omega) := \omega \cdot \mathrm{diag}(-K,\ldots,K), \quad \omega = \frac{2\pi}{T} \tag{10}$$

and $\Omega := I_N \otimes \bar{\Omega}$. HB solves the algebraic system of equations

$$\begin{aligned} F(V) &= P\,i(P^{-1}V) + j\,\Omega(\omega)\cdot P\,q(P^{-1}V) + P\,b \\ &= I(V) + j\,\Omega(\omega)\cdot Q(V) + B = 0 \end{aligned} \tag{11}$$

for the unknown vector of Fourier coefficients V. Equation (**??**) is an algebraic system of equations $F : \mathbb{C}^{(2K+1)N} \to \mathbb{C}^{(2K+1)N}$ for the unknowns V which can be solved by Newton-like methods [?, **?**]. The evaluation of the Jacobian is given i.e. in [?]. HB can be generalized to quasiperiodic steady states. For more details see i.e. [?].

3 Spline interpolation

In the cases of sharp transients the convergence of the Fourier series is poor and only guaranteed in the L_2 norm.

Alternatively, in this section cubic and exponential splines are considered as an alternative to Fourier basis functions. Unlike Fourier series, the spline basis functions are only locally defined, therefore the approximation of sharp transients is significantly improved. We restrict here to an equidistant discretization. In that case the coding of the spline basis is simple when a HB simulator exists. Circuit designers are mainly interested in the spectrum of the waveforms. We show here further, that the spectrum can be directly evaluated from the coefficients of the spline approximation.

3.1 Cubic spline interpolation

Definition 1 ([?]). Let $t_0 < t_1 < \cdots < t_n$ be an ordered sequence of collocation points. The B-splines $\hat{y}_{ik}(t)$ of order k for $k = 1,\ldots,n$ and $i = 1,\ldots,n-k$ are defined recursively by

$$\hat{y}_{i1}(t) := \begin{cases} 1 & \text{if} \quad t_i \le t < t_{i+1} \\ 0 & \text{elsewhere} \end{cases}, \; \hat{y}_{ik}(t) := \frac{t - t_i}{t_{i+k-1} - t_i}\,\hat{y}_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}}\,\hat{y}_{i+1,k-1}(t)$$

The cubic spline solves the variational problem [**?**, ?]

$$E[y] = \sum_{i=0}^{n-1} \int_{t_i}^{t_i+\Delta t} \big(\ddot{y}(t)\big)^2 \mathrm{d}t \tag{12}$$

We denote as $t_{i+1} - t_i =: \Delta t$ the grid spacing. The collocation points coincide therefore with HB based on a Fourier expansion. This eases the implementation as shown below.

The periodic waveform $x(t) = x(t+T)$ is approximated by a linear combination of weighted and shifted basis functions $\hat{y}(t)$, the shifts being integer multiples of the grid spacing $t_l = l\cdot\Delta t,\ l = 0,1,\ldots$

$$y(t) = \sum_{l=0}^{2K} \hat{Y}(l) \cdot \hat{y}(t - t_l) \tag{13}$$

The unknown coefficients $\hat{Y}(l)$ are uniquely calculated by requiring that the error vanishes at the collocation points and the periodicity constraint of the signal waveform. One obtains the system of equations with $x(l) := x(t_l)$

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{6} & & & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & & & \frac{1}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \hat{Y}(0) \\ \vdots \\ \hat{Y}(2K) \end{bmatrix} = \begin{bmatrix} x(0) \\ \vdots \\ x(2K) \end{bmatrix} \tag{14}$$

The coefficients matrix is circulant, representing the periodicity constraint, the eigenvectors are therefore the column vectors of the DFT $z_k = \left[\exp\left(-jk \cdot \frac{2\pi}{2K+1} K\right), \ldots, \exp\left(jk \cdot \frac{2\pi}{2K+1} K\right)\right]^T$ with corresponding eigenvalues $\lambda_k = \frac{2}{3} + \frac{1}{3} \cos\left(k \cdot \frac{2\pi}{2K+1}\right)$, $-K \leq k \leq K$. For solving the underlying DAE, time derivatives of the approximation at the grid points are required. Introducing the operator ∇ of the time-derivatives at the collocation points and the coefficient matrix of the DFT $\mathscr{F}$,

$$j\,\bar{\Omega} = \mathscr{F}\ \nabla\ \mathscr{F}^{-1}$$

holds. The matrix $\bar{\Omega}$ is again a diagonal matrix

$$\bar{\Omega} = \text{diag}\ (\omega(-K), \ldots, \omega(k), \ldots, \omega(K)) \tag{15}$$

The diagonal elements $\omega(k)$ are obtained from the eigenvalues by

$$\omega(k) = \left(\frac{2K+1}{2\pi}\right) \omega_0 \frac{3 \sin\left(k \cdot \frac{2\pi}{2K+1}\right)}{2 + \cos\left(k \cdot \frac{2\pi}{2K+1}\right)} \tag{16}$$

The additional coding load for (**??**, **??**) is marginal, if a HB simulator exists because the sparsity structure of the $\bar{\Omega}$ matrix of the HB and spline methods are identical.

The coefficients $\hat{Y}(l)$ of the spline basis are of minor interest for circuit designers. Instead, the Fourier spectrum is of superior interest. From the spline approximation (**??**) one can calculate the Fourier spectrum exactly. Based on the DFT of the collocation points X_k one gets the spectrum of the spline approximation by

$$\frac{Y_k}{X_k} = \frac{12}{z^4} \frac{(1 - \cos z)^2}{2 + \cos z}, \quad z = \frac{2\pi k}{2K+1}, \quad k \neq 0 \tag{17}$$

The fig. **??**(a) illustrates the simulated limit-cycle for a Colpitt oscillator. 128 equidistant gridpoints are taken and fig. **??**(b) shows the difference signal between the HB and the spline solution.

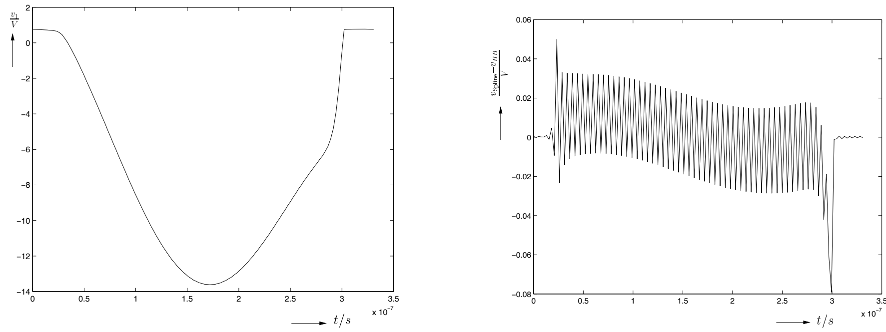

Fig. 1 (a) Calculated limit cycle by a cubic spline approximation, (b) Comparison between the numerical solutions by cubic spline and trigonometric basis function.

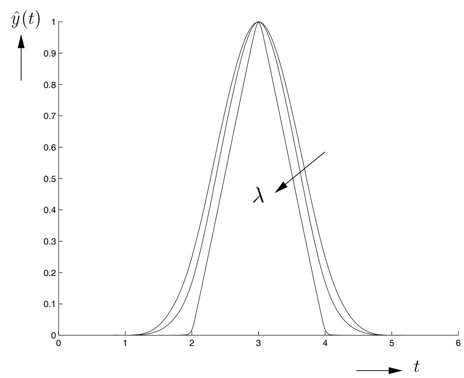

Fig. 2 Exponential spline basis functions for the parameter set $\lambda = 10^{-0,5}$, $10^{0,5}$ and $\lambda = 10^{1,5}$.

3.2 Exponential splines

The exponential splines [?, ?] can be fit to the specific interpolation problem by a free parameter λ. They solve the variational problem

$$E[y] = \sum_{i=0}^{n-1} \int_{t_i}^{t_i+\Delta t} \left[\big(\ddot{y}(t)\big)^2 + \frac{\lambda^2}{\Delta t^2} \big(\dot{y}(t)\big)^2 \right] \mathrm{d}t \tag{18}$$

The fig. **??** depicts the exponential spline for different parameter values of λ.

The next steps are formally identical to the treatment of the cubic splines and are only summarized for brevity. One gets a diagonal matrix representing the collection of derivatives to time in the frequency domain by $j\,\bar{\Omega} = \mathscr{F}\,\nabla\,\mathscr{F}^{-1} = j\,\mathrm{diag}\,\{\omega(-K),\ldots,\omega(k),\ldots,\omega(K)\}$ wherein the diagonal elements are expressed by

$$\omega(k) = 2\left(\frac{2K+1}{2\pi}\right)\omega_0\,\frac{1}{\lambda^2}\,\frac{[\cosh\lambda - 1]\,\sin\left(\frac{2\pi k}{2K+1}\right)}{c + \frac{2}{\lambda^3}\,[\sinh\lambda - \lambda]\,\cos\left(\frac{2\pi k}{2K+1}\right)} \tag{19}$$

3.3 Cubic spline for multitone steady state analysis

For keeping the derivation as simple as possible only the 2-tone case is considered here. The generalization to d-quasiperiodic steady states is simple, making use of the partial DAE formulation (**??**).

A two-dimensional spline basis function can be written as the product of one-dimensional basis functions, i.e.

$$\hat{y}(t_1,t_2) = \hat{y}(t_1)\cdot\hat{y}(t_2) \tag{20}$$

Similarly to (**??**) one gets a system of equations for the coefficients $\hat{Y}_{l_1 l_2}$ of the spline interpolation

$$y(t_1,t_2) = \sum_{l_1,l_2\subset\mathbb{Z}} \hat{Y}_{l_1 l_2}\cdot\hat{y}(t_1 - t_{l_1}, t_2 - t_{l_2}) \tag{21}$$

with pre-supposed periodic conditions in t_1 and t_2 and grid-points t_{l_1} and t_{l_2}.

The fig. **??** (left) illustrates how the coefficients of the matrix are obtained. Due to the periodicity in t_1 and t_2, the coefficient matrix is a hierarchical or nested circulant matrix, i.e. any block of the circulant matrix is itself a circulant matrix.

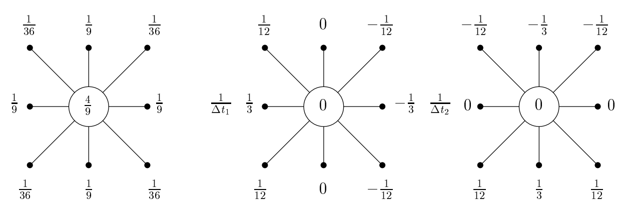

Fig. 3 Molecule for evaluating the coefficients of the spline interpolation (left) and its partial derivatives.

Further, the partial DAE requires the sum of partial derivatives, which must be calculated for the spline interpolation at the collocation points. From a similar derivation (fig. **??**), one obtains the entries for the $\bar{\Omega}$ matrix

$$\omega(k_1,k_2) = \frac{\mu(k_1,k_2)}{\lambda(k_1,k_2)} = \frac{3}{\Delta t_1}\,\frac{\sin\left(\frac{2\pi k_1}{2K_1+1}\right)}{2+\cos\left(\frac{2\pi k_1}{2K_1+1}\right)} + \frac{3}{\Delta t_2}\,\frac{\sin\left(\frac{2\pi k_2}{2K_2+1}\right)}{2+\cos\left(\frac{2\pi k_2}{2K_2+1}\right)} \tag{22}$$

Please note that for multitone HB one gets $\omega(k_1,k_2) = k_1\,\omega_1 + k_2\,\omega_2$.

Again, the spectrum of the spline approximation can be evaluated from the trapezoidal method. The derivation is similar to the periodic case.

Conclusions

Cubic and exponential spline bases are an interesting alternative for simulating periodic and quasiperiodic steady states when sharp transients occur in the waveforms. The implementation effort is negligible when a code for the Harmonic Balance technique is available. The Fourier spectrum can easily be calculated from the spline approximation which is very important for electronic engineers.

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