Hybrid Analysis of Nonlinear Time-Varying Circuits Providing DAEs with Index at Most One

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Abstract Commercial packages for transient circuit simulation are often based on the modified nodal analysis (MNA) which allows an automatic setup of model equations and requires a nearly minimal number of variables. However, it may lead to differential-algebraic equations (DAEs) with higher index. Here, we present a hybrid analysis for nonlinear time-varying circuits leading to DAEs with index at most one. This hybrid analysis is based merely on the network topology, which possibly leads to an automatic setup of the hybrid equations from netlists. Moreover, we prove that the minimum index of the DAE arising from the hybrid analysis never exceeds the index from MNA. As a positive side effect, the number of equations from the hybrid analysis is always no greater than that one from MNA. This suggests that the hybrid analysis is superior to MNA in numerical accuracy and computational effort.

1 Introduction

When modelling electric circuits for transient simulation, one has to regard Kirchhoff's laws for the network and the constitutive equations for the different types of network elements. They are originally based on the branch voltages and the branch

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currents existing in the network. They form the basis for all modelling approaches as for instance the popular modified nodal analysis (MNA).

Concerning the huge number of variables involved (all branch voltages and branch currents), one is interested in a reduced system reflecting the complete circuit behaviour that can be generated automatically. Whereas MNA focuses on a description depending mainly on nodal potentials, the hybrid analysis approach [1] here employs certain branch voltages and branch currents obtained from a construction of a particular *normal tree*.

A normal tree is a tree containing all independent voltage sources, no independent current sources, a maximal number of capacitive branches, and a minimal number of inductive branches. Normal trees have already been used in [2] for state approaches for linear RLC networks. The results have been extended in [3] for linear circuits containing ideal transformers, nullors, independent/dependent sources, resistors, inductors, capacitors, and, under a topological restriction, gyrators.

The hybrid analysis is a common generalization of the loop analysis and the cutset analysis. Kron [4] proposed the hybrid analysis in 1939, and Amari [5] and Branin [6] developed it further in 1960s. In contrast to MNA, the hybrid analysis retains flexibility in the selection of a normal tree, which can be exploited to find a model description that reduces the numerical difficulties.

The differential-algebraic equations (DAEs) arising from the hybrid analysis are called the *hybrid equations*. Recently, the analysis of the *index* of the hybrid equations has been developed. For linear time-invariant RLC circuits, it is shown in [7] that the index of the hybrid equations never exceeds one, while MNA often results in a DAE with index two. Moreover, [7] gives a structural characterization of circuits with index zero. For linear time-invariant electric circuits which may contain dependent voltage/current sources, an algorithm for finding an optimal hybrid analysis which minimizes the index of the hybrid equations was proposed in [8].

For nonlinear time-varying circuits, this paper shows that the index of the hybrid equations is at most one, and gives a structural characterization for the index being zero, which is an extension of the results in [7]. By this structural characterization, we prove that the minimum index of the hybrid equations does not exceed the index of the DAE arising from MNA (cf. [9–11]). Here, we follow the hybrid analysis approach in [8] but use projection techniques (cf. [10]) in order to prove the index results for general nonlinear time-varying circuit systems.

The organization of this paper is as follows. In Section 2, we describe nonlinear time-varying circuits. We present the procedure of the hybrid analysis in Section 3. We analyze the hybrid equation system in Section 4, and characterize its index in Section 5. All the technical proofs omitted in this paper can be found in [12].

2 Nonlinear Time-Varying Circuits

Here, we consider nonlinear time-varying circuits composed of resistors, conductors, inductors, capacitors, and voltage/current sources.

We denote the vector of branch currents by $\mathbf{i}$, and the vector of branch voltages by $\mathbf{u}$. The vector of currents through independent voltage sources, independent current sources, capacitors, inductors, resistors, conductors, controlled current sources, and controlled voltage sources are denoted by $\mathbf{i}_V$, $\mathbf{i}_J$, $\mathbf{i}_C$, $\mathbf{i}_L$, $\mathbf{i}_R$, $\mathbf{i}_G$, $\mathbf{i}_{S_J}$, and $\mathbf{i}_{S_V}$. Similarly, the vector of voltages are denoted by $\mathbf{u}_V$, $\mathbf{u}_J$, $\mathbf{u}_C$, $\mathbf{u}_L$, $\mathbf{u}_R$, $\mathbf{u}_G$, $\mathbf{u}_{S_J}$, and $\mathbf{u}_{S_V}$. The physical characteristics of elements determine *constitutive equations*. Independent voltage and current sources simply read as

$$\mathbf{u}_V = \mathbf{v}_s(t) \quad \text{and} \quad \mathbf{i}_J = \mathbf{j}_s(t). \tag{1}$$

Capacitors and inductors can be modelled by

$$\mathbf{i}_C = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}(\mathbf{u}_C,t) \quad \text{and} \quad \mathbf{u}_L = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{i}_L,t). \tag{2}$$

Moreover, we assume that conductors and resistors are described by $\mathbf{i}_G = \mathbf{g}(\mathbf{u}_G,t)$ and $\mathbf{u}_R = \mathbf{r}(\mathbf{i}_R,t)$. Finally, let the controlled sources be given in the form of $\mathbf{i}_{S_J} = \gamma(\mathbf{i}_{S_V},\mathbf{u}_{S_J},t)$ and $\mathbf{u}_{S_V} = \rho(\mathbf{i}_{S_V},\mathbf{u}_{S_J},t)$.

A square matrix U is called *positive definite* if $\mathbf{x}^\top U\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$. In this paper, we assume the following conditions.

Assumption 1 *The capacitance matrix C, the conductance matrix G, the resistance matrix R, the inductance matrix L, and the controlled source matrix S given by*

$$C = \frac{\partial \mathbf{q}}{\partial \mathbf{u}_C}, \quad G = \frac{\partial \mathbf{g}}{\partial \mathbf{u}_G}, \quad R = \frac{\partial \mathbf{r}}{\partial \mathbf{i}_R}, \quad L = \frac{\partial \phi}{\partial \mathbf{i}_L}, \quad \textit{and} \quad S = \begin{pmatrix} \frac{\partial \rho}{\partial \mathbf{i}_{S_V}} & \frac{\partial \rho}{\partial \mathbf{u}_{S_J}} \\ \frac{\partial \gamma}{\partial \mathbf{i}_{S_V}} & \frac{\partial \gamma}{\partial \mathbf{u}_{S_J}} \end{pmatrix}$$

are all positive definite.[1]

Introducing $\mathbf{u}_Y := \begin{pmatrix} \mathbf{u}_G \\ \mathbf{u}_{S_J} \end{pmatrix}$, $\mathbf{u}_Z := \begin{pmatrix} \mathbf{u}_R \\ \mathbf{u}_{S_V} \end{pmatrix}$, $\mathbf{i}_Y := \begin{pmatrix} \mathbf{i}_G \\ \mathbf{i}_{S_J} \end{pmatrix}$, $\mathbf{i}_Z := \begin{pmatrix} \mathbf{i}_R \\ \mathbf{i}_{S_V} \end{pmatrix}$, $\mathbf{f}(\mathbf{i}_Z,\mathbf{u}_Y,t) := \begin{pmatrix} \mathbf{g}(\mathbf{u}_G,t) \\ \gamma(\mathbf{i}_{S_V},\mathbf{u}_{S_J},t) \end{pmatrix}$, and $\mathbf{h}(\mathbf{i}_Z,\mathbf{u}_Y,t) := \begin{pmatrix} \mathbf{r}(\mathbf{i}_R,t) \\ \rho(\mathbf{i}_{S_V},\mathbf{u}_{S_J},t) \end{pmatrix}$, we find

$$\mathbf{i}_Y = \mathbf{f}(\mathbf{i}_Z,\mathbf{u}_Y,t), \quad \mathbf{u}_Z = \mathbf{h}(\mathbf{i}_Z,\mathbf{u}_Y,t) \tag{3}$$

and the matrix $\begin{pmatrix} \frac{\partial \mathbf{h}}{\partial \mathbf{i}_Z} & \frac{\partial \mathbf{h}}{\partial \mathbf{u}_Y} \\ \frac{\partial \mathbf{f}}{\partial \mathbf{i}_Z} & \frac{\partial \mathbf{f}}{\partial \mathbf{u}_Y} \end{pmatrix}$ to be positive definite because of Assumption 1.

Let $\Gamma = (W,E)$ be the connected network graph with vertex set W and edge set E. An edge in Γ corresponds to a branch that contains one element in the circuit. For a consistent model description, Γ contains no cycles consisting of independent

[1] Assuming the controlled source matrix S to be positive definite is very restrictive and usually not fulfilled when controlled sources are considered alone. However, controlled sources are often used to describe certain transistor behaviour. Considering the whole static behavior of a transistor (e.g. including bulk resistances) as a controlled source may lead to a positive definite matrix S.

voltage sources only and no cutsets consisting of independent current sources only. We split E into E_y and E_z, i.e., $E_y \cup E_z = E$ and $E_y \cap E_z = \emptyset$. A partition (E_y, E_z) is called an *admissible partition*, if E_y includes all the independent voltage sources, all the capacitors, all the conductors as well as all the controlled current sources, and E_z includes all the independent current sources, all the inductors, all the resistors as well as all the controlled voltage sources.

We call a spanning tree T of Γ a *reference tree* if T contains all the edges of the independent voltage sources, no edges of the independent current sources, and as many edges in E_y as possible. Note that a reference tree T may contain some edges in E_z. A reference tree is called *normal* if it contains as many edges corresponding to capacitors and as few edges corresponding to inductors as possible. The cotree of T is denoted by $\overline{T} = E \setminus T$. The hybrid equations are determined by an admissible partition (E_y, E_z) and a reference tree T, which is not necessarily normal. For the sake of simplicity, we adopt a normal reference tree throughout this paper.

With respect to a normal reference tree T, we further split $\mathbf{i}$ and $\mathbf{u}$ into

$$\mathbf{i} = (\mathbf{i}_V, \mathbf{i}_C^\tau, \mathbf{i}_Y^\tau, \mathbf{i}_Z^\tau, \mathbf{i}_L^\tau, \mathbf{i}_C^\lambda, \mathbf{i}_Y^\lambda, \mathbf{i}_Z^\lambda, \mathbf{i}_L^\lambda, \mathbf{i}_J)^\top \text{ and } \mathbf{u} = (\mathbf{u}_V, \mathbf{u}_C^\tau, \mathbf{u}_Y^\tau, \mathbf{u}_Z^\tau, \mathbf{u}_L^\tau, \mathbf{u}_C^\lambda, \mathbf{u}_Y^\lambda, \mathbf{u}_Z^\lambda, \mathbf{u}_L^\lambda, \mathbf{u}_J)^\top,$$

where the superscripts τ and λ designate the tree T and the cotree $\overline{T}$. With respect to a normal reference tree T, the vector valued function $\mathbf{f}$ is also split into $\mathbf{f}^\tau$ and $\mathbf{f}^\lambda$. This means $\mathbf{i}_Y^\tau = \mathbf{f}^\tau(\mathbf{i}_Z, \mathbf{u}_Y, t)$ and $\mathbf{i}_Y^\lambda = \mathbf{f}^\lambda(\mathbf{i}_Z, \mathbf{u}_Y, t)$. Similarly, we split $\mathbf{h}$, $\mathbf{q}$, and ϕ.

By the definition of a normal reference tree, the *fundamental cutset matrix* K is given by

$$K = \begin{array}{c} \begin{matrix} \mathbf{i}_V & \mathbf{i}_C^\tau & \mathbf{i}_Y^\tau & \mathbf{i}_Z^\tau & \mathbf{i}_L^\tau & \mathbf{i}_C^\lambda & \mathbf{i}_Y^\lambda & \mathbf{i}_Z^\lambda & \mathbf{i}_L^\lambda & \mathbf{i}_J \end{matrix} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 & A_{VC} & A_{VY} & A_{VZ} & A_{VL} & A_{VJ} \\ 0 & I & 0 & 0 & 0 & A_{CC} & A_{CY} & A_{CZ} & A_{CL} & A_{CJ} \\ 0 & 0 & I & 0 & 0 & 0 & A_{YY} & A_{YZ} & A_{YL} & A_{YJ} \\ 0 & 0 & 0 & I & 0 & 0 & 0 & A_{ZZ} & A_{ZL} & A_{ZJ} \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & A_{LL} & A_{LJ} \end{pmatrix} \end{array}.$$

Then *Kirchhoff's current law* (KCL) may be written as $K\mathbf{i} = \mathbf{0}$. *Kirchhoff's voltage law* (KVL) provides $K^\perp \mathbf{u} = \mathbf{0}$ with $K^\perp$ being the *fundamental loop matrix*

$$K^\perp = \begin{array}{c} \begin{matrix} \mathbf{u}_V & \mathbf{u}_C^\tau & \mathbf{u}_Y^\tau & \mathbf{u}_Z^\tau & \mathbf{u}_L^\tau & \mathbf{u}_C^\lambda & \mathbf{u}_Y^\lambda & \mathbf{u}_Z^\lambda & \mathbf{u}_L^\lambda & \mathbf{u}_J \end{matrix} \\ \begin{pmatrix} -A_{VC}^\top & -A_{CC}^\top & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ -A_{VY}^\top & -A_{CY}^\top & -A_{YY}^\top & 0 & 0 & 0 & I & 0 & 0 & 0 \\ -A_{VZ}^\top & -A_{CZ}^\top & -A_{YZ}^\top & -A_{ZZ}^\top & 0 & 0 & 0 & I & 0 & 0 \\ -A_{VL}^\top & -A_{CL}^\top & -A_{YL}^\top & -A_{ZL}^\top & -A_{LL}^\top & 0 & 0 & 0 & I & 0 \\ -A_{VJ}^\top & -A_{CJ}^\top & -A_{YJ}^\top & -A_{ZJ}^\top & -A_{LJ}^\top & 0 & 0 & 0 & 0 & I \end{pmatrix} \end{array}.$$

3 Hybrid Analysis

In this section, we describe the procedure of the hybrid analysis. The idea is to use all constitutive equations such that the equations $K\mathbf{i} = \mathbf{0}$ and $K^\perp \mathbf{u} = \mathbf{0}$ provide a system depending on $\mathbf{u}_C^\tau$, $\mathbf{u}_Y^\tau$, $\mathbf{i}_Z^\lambda$, and $\mathbf{i}_L^\lambda$ only. The details are described in [12]. The second and third line of $K\mathbf{i} = \mathbf{0}$ as well as the third and fourth line of $K^{\perp}\mathbf{u} = \mathbf{0}$ provide us the *hybrid equations* (or *hybrid equation system*)

$$\begin{aligned}
-A_{CZ}^{\top}\mathbf{u}_C^{\tau} - A_{YZ}^{\top}\mathbf{u}_Y^{\tau} - A_{ZZ}^{\top}\mathbf{h}^{\tau} + \mathbf{h}^{\lambda} &= A_{VZ}^{\top}\mathbf{v}_s(t),\\
-A_{CL}^{\top}\mathbf{u}_C^{\tau} - A_{YL}^{\top}\mathbf{u}_Y^{\tau} - A_{ZL}^{\top}\mathbf{h}^{\tau} - A_{LL}^{\top}\frac{\mathrm{d}}{\mathrm{d}t}\phi^{\tau} + \frac{\mathrm{d}}{\mathrm{d}t}\phi^{\lambda} &= A_{VL}^{\top}\mathbf{v}_s(t),\\
A_{CY}\mathbf{f}^{\lambda} + A_{CZ}\mathbf{i}_Z^{\lambda} + A_{CL}\mathbf{i}_L^{\lambda} + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}^{\tau} + A_{CC}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}^{\lambda} &= -A_{CJ}\mathbf{j}_s(t),\\
\mathbf{f}^{\tau} + A_{YY}\mathbf{f}^{\lambda} + A_{YZ}\mathbf{i}_Z^{\lambda} + A_{YL}\mathbf{i}_L^{\lambda} &= -A_{YJ}\mathbf{j}_s(t),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{q} &= \mathbf{q}(\mathbf{u}_C^{\tau}, A_{VC}^{\top}\mathbf{v}_s(t) + A_{CC}^{\top}\mathbf{u}_C^{\tau}, t), \quad \phi \;=\; \phi(-A_{LL}\mathbf{i}_L^{\lambda} - A_{LJ}\mathbf{j}_s(t), \mathbf{i}_L^{\lambda}, t),\\
\mathbf{f} &= \mathbf{f}(-A_{ZZ}\mathbf{i}_Z^{\lambda} - A_{ZL}\mathbf{i}_L^{\lambda} - A_{ZJ}\mathbf{j}_s(t), \mathbf{i}_Z^{\lambda}, \mathbf{u}_Y^{\tau}, A_{VY}^{\top}\mathbf{v}_s(t) + A_{CY}^{\top}\mathbf{u}_C^{\tau} + A_{YY}^{\top}\mathbf{u}_Y^{\tau}, t),\\
\mathbf{h} &= \mathbf{h}(-A_{ZZ}\mathbf{i}_Z^{\lambda} - A_{ZL}\mathbf{i}_L^{\lambda} - A_{ZJ}\mathbf{j}_s(t), \mathbf{i}_Z^{\lambda}, \mathbf{u}_Y^{\tau}, A_{VY}^{\top}\mathbf{v}_s(t) + A_{CY}^{\top}\mathbf{u}_C^{\tau} + A_{YY}^{\top}\mathbf{u}_Y^{\tau}, t).
\end{aligned}$$

The procedure of the hybrid analysis is as follows.

1. The values of $\mathbf{u}_V$ and $\mathbf{i}_J$ are obvious from (1).
2. Compute the values of $\mathbf{i}_Z^{\lambda}$, $\mathbf{i}_L^{\lambda}$ and $\mathbf{u}_C^{\tau}$, $\mathbf{u}_Y^{\tau}$ by solving the hybrid equations.
3. Compute the values of $\mathbf{i}_Z^{\tau}$, $\mathbf{i}_L^{\tau}$ from the fourth and fifth line of $K\mathbf{i} = \mathbf{0}$ (KCL) and $\mathbf{u}_C^{\lambda}$, $\mathbf{u}_Y^{\lambda}$ from the first and second line of $K^{\perp}\mathbf{u} = \mathbf{0}$ (KVL) by substituting the values obtained in Steps 1–2.
4. Compute the values of $\mathbf{u}_Z^{\tau}$, $\mathbf{u}_Z^{\lambda}$, $\mathbf{u}_L^{\tau}$, $\mathbf{u}_L^{\lambda}$, and $\mathbf{i}_C^{\tau}$, $\mathbf{i}_C^{\lambda}$, $\mathbf{i}_Y^{\tau}$, $\mathbf{i}_Y^{\lambda}$ by substituting the values obtained in Steps 1–3 into (2) and (3).
5. Compute the values of $\mathbf{i}_V$ and $\mathbf{u}_J$ by substituting the values obtained in Steps 1–4 into the first line of KCL and the fifth line of KVL.

All operations in Steps 3–5 are substitutions and differentiations of the obtained solutions. Consequently, the numerical difficulty is determined by the index of the hybrid equation system. Higher index variables as known from MNA do not appear in the hybrid equation system. In this paper, we prove that the hybrid equation system has index at most one. The proof relies on the *tractability index* concept for DAEs with the use of a projector based analysis.

4 Hybrid Equations with Properly Stated Leading Term

Consider a DAE in the form of

$$A\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{d}(\mathbf{x}(t),t) + \mathbf{b}(\mathbf{x}(t),t) = \mathbf{0}. \tag{4}$$

Let A be an $m \times n$ matrix. We define $D(\mathbf{x},t) := \dfrac{\partial \mathbf{d}(\mathbf{x},t)}{\partial \mathbf{x}}$, $B(\mathbf{x},t) := \dfrac{\partial \mathbf{b}(\mathbf{x},t)}{\partial \mathbf{x}}$, and $M(\mathbf{x},t) := AD(\mathbf{x},t)$. A matrix P satisfying $P^2 = P$ is called a *projector*.

Definition 1 ([13, Definition 2.1]). The equation (4) is a *DAE with properly stated leading term* if the size of $D(\mathbf{x},t)$ is $n \times m$, $\ker A \oplus \operatorname{im} D(\mathbf{x},t) = \mathbb{R}^n$ holds for all $\mathbf{x}$ and t from the definition domain, and there is an $n \times n$ projector function $P(t)$ continuously differentiable with respect to t such that $\ker P(t) = \ker A$, $\operatorname{im} P(t) = \operatorname{im} D(\mathbf{x},t)$, and $\mathbf{d}(\mathbf{x},t) = P(t)\mathbf{d}(\mathbf{x},t)$.

A DAE with properly stated leading term (4) arises in circuit simulation via analysis methods such as MNA [14]. A DAE with properly stated leading term was first introduced in [15]. The analysis of such DAEs has been developed in [14, 16–19].

Obviously, the DAE (4) represents a regular ODE if and only if the matrix $M(\mathbf{x},t)$ is nonsingular for all $\mathbf{x}$ and t of the definition domain. In this case we say that the DAE (4) has index 0. In the case of a singular matrix $M(\mathbf{x},t)$ for all $\mathbf{x}$ and t, the DAE (4) contains algebraic equations. Furthermore, one may have to differentiate certain part of the system to get a solution. A simple criteria for the absence of this problem is given by the tractability index 1 condition (see [13], Theorem 4.3).

Definition 2 ([13, Definition 3.3]). The DAE (4) is *regular with index 1* on their definition domain if $M(\mathbf{x},t)$ is singular and $\ker D(\mathbf{x},t) \cap \{\mathbf{z} \in \mathbb{R}^m | \ B(\mathbf{x},t)\mathbf{z} \in \operatorname{im} M(\mathbf{x},t)\} = \{\mathbf{0}\}$ for all $(\mathbf{x},t)$ of the definition domain.

Remark 1 ([20, Remark 4.6]). A DAE (4) is regular with index 1 if and only if the matrix $M(\mathbf{x},t) + B(\mathbf{x},t)Q(\mathbf{x},t)$ is nonsingular for all $\mathbf{x}$ and t with a projector $Q(\mathbf{x},t)$ satisfying $\operatorname{im} Q(\mathbf{x},t) = \ker M(\mathbf{x},t)$.

We rewrite the hybrid equation system as a DAE with properly stated leading term. A *reflexive generalized inverse* of a matrix A is a matrix A^- which satisfies $AA^-A = A$ and $A^-AA^- = A^-$. Let us define

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -A_{LL}^\top & I & 0 & 0 \\ 0 & 0 & I & A_{CC} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{d}(\mathbf{x},t) = A^-A \begin{pmatrix} \phi^\tau(-A_{LL}\mathbf{i}_L^\lambda - A_{LJ}\mathbf{j}_s(t), \mathbf{i}_L^\lambda, t) \\ \phi^\lambda(-A_{LL}\mathbf{i}_L^\lambda - A_{LJ}\mathbf{j}_s(t), \mathbf{i}_L^\lambda, t) \\ \mathbf{q}^\tau(\mathbf{u}_C^\tau, A_{VC}^\top\mathbf{v}_s(t) + A_{CC}^\top\mathbf{u}_C^\tau, t) \\ \mathbf{q}^\lambda(\mathbf{u}_C^\tau, A_{VC}^\top\mathbf{v}_s(t) + A_{CC}^\top\mathbf{u}_C^\tau, t) \end{pmatrix},$$

$$\mathbf{x}(t) = \begin{pmatrix} \mathbf{i}_Z^\lambda \\ \mathbf{i}_L^\lambda \\ \mathbf{u}_C^\tau \\ \mathbf{u}_Y^\tau \end{pmatrix}, \quad \mathbf{b}(\mathbf{x},t) = \begin{pmatrix} -A_{VZ}^\top\mathbf{v}_s(t) - A_{CZ}^\top\mathbf{u}_C^\tau - A_{YZ}^\top\mathbf{u}_Y^\tau - A_{ZZ}^\top\mathbf{h}^\tau + \mathbf{h}^\lambda \\ -A_{VL}^\top\mathbf{v}_s(t) - A_{CL}^\top\mathbf{u}_C^\tau - A_{YL}^\top\mathbf{u}_Y^\tau - A_{ZL}^\top\mathbf{h}^\tau \\ A_{CY}\mathbf{f}^\lambda + A_{CZ}\mathbf{i}_Z^\lambda + A_{CL}\mathbf{i}_L^\lambda + A_{CJ}\mathbf{j}_s(t) \\ \mathbf{f}^\tau + A_{YY}\mathbf{f}^\lambda + A_{YZ}\mathbf{i}_Z^\lambda + A_{YL}\mathbf{i}_L^\lambda + A_{YJ}\mathbf{j}_s(t) \end{pmatrix}.$$

This gives the hybrid equation system in the form of (4). Under Assumption 1, the hybrid equation system (4) is shown to be a DAE with properly stated leading term.

5 Index of Hybrid Equations

In this section, we show that the index of the hybrid equations is at most one, and give a structural criteria for hybrid equations with index zero. We now introduce the *Resistor-Acyclic condition* for admissible partition (E_y, E_z).

[Resistor-Acyclic condition]

- Each conductor and controlled current source in E_y belongs to a cycle consisting of independent voltage sources, capacitors, and itself.
- Each resistor and controlled voltage source in E_z belongs to a cutset consisting of inductors, independent current sources, and itself.

Consider the graph $\tilde{\Gamma}$ obtained from $\Gamma = (W, E)$ by contracting all edges of independent voltage sources and capacitors and deleting all edges of inductors and independent current sources. The Resistor-Acyclic condition means that $\tilde{\Gamma}$ is acyclic [7].

Theorem 1. *Under Assumption 1, the index of the hybrid equations is at most one for any admissible partition (E_y, E_z) and normal reference tree T. Moreover, the index is zero if and only if an admissible partition (E_y, E_z) satisfies the Resistor-Acyclic condition.*

Here we present only a sketch of the proof. Details are given in [12]. Computation of the matrix $M(\mathbf{x},t) + B(\mathbf{x},t)Q(\mathbf{x},t)$ leads to (omitting the arguments)

$$M + BQ = \begin{pmatrix} B_Z & 0 & 0 & -A_{YZ}^\top + B_H \\ * & M_L & 0 & * \\ * & 0 & M_C & * \\ A_{YZ} + B_F & 0 & 0 & B_Y \end{pmatrix} \quad \text{for} \quad Q = \begin{pmatrix} I & & & \\ & 0 & & \\ & & 0 & \\ & & & I \end{pmatrix}.$$

Here, M_L and M_C are nonsingular and $\begin{pmatrix} B_Z & -A_{YZ}^\top + B_H \\ A_{YZ} + B_F & B_Y \end{pmatrix} = \begin{pmatrix} 0 & -A_{YZ}^\top \\ A_{YZ} & 0 \end{pmatrix} + \begin{pmatrix} B_Z & B_H \\ B_F & B_Y \end{pmatrix}$ is a sum of a positive semidefinite and a positive definite matrix. These properties imply $M + BQ$ to be nonsingular. Since $AD(\mathbf{x},t) = A \begin{pmatrix} L & 0 \\ 0 & C \end{pmatrix} A^\top$ holds, also the second statement of Theorem 1 is clear.

By Theorem 1, we can prove that the minimum index of the hybrid equations never exceeds the index of the DAE arising from MNA for nonlinear time-varying circuits without controlled voltage/current sources.

Remark 2. A simple algorithm for finding the optimal admissible partition is given in [7]. See [7, Examples 4.13–4.14] for circuit examples, which trace the procedure of the hybrid analysis and make comparisons between the hybrid analysis and MNA.

Remark 3. For nonlinear time-varying circuits composed of resistors (all modelled as conductances), inductors, capacitors, and voltage/current sources, the dimension of the hybrid equation system is no greater than that one for the MNA system. This is because $\dim(\mathbf{u}_C^\tau, \mathbf{u}_Y^\tau) < n$ for n being the number of nodes of the circuit, $\dim \mathbf{i}_L^\lambda$ is not greater than the number of inductors in the system, and $\dim \mathbf{i}_Z^\lambda$ is not greater than the number of (controlled) voltage sources of the system.

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